Presentation 7.3a: Multiple linear regression

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1. Theory

1.1. Multiple Linear Regression

1.1.1. Additive model

$$growth = intercept + temperature + nitrogen$$

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_j x_{ij} + \epsilon_i$$

OR

$$y_i = \beta_0 + \sum_{j=1:n}^{N} \beta_j x_{ji} + \epsilon_i$$

1.2. Multiple Linear Regression

1.2.1. Additive model

$$growth = intercept + temperature + nitrogen$$

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_j x_{ij} + \epsilon_i$$

- effect of one predictor holding the other(s) constant

1.3. Multiple Linear Regression

1.3.1. Additive model

$growth = intercept + temperature + nitrogen$

$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_j x_{ij} + \epsilon_i$

Y	X1	X2
3	22.7	0.9
2.5	23.7	0.5
6	25.7	0.6
5.5	29.1	0.7
9	22	0.8
8.6	29	1.3
12	29.4	1

1.4. Multiple Linear Regression

1.4.1. Additive model

$$
\begin{aligned}
3 &= \beta_0 + (\beta_1 \times 22.7) + (\beta_2 \times 0.9) + \varepsilon_1 \\
2.5 &= \beta_0 + (\beta_1 \times 23.7) + (\beta_2 \times 0.5) + \varepsilon_2 \\
6 &= \beta_0 + (\beta_1 \times 25.7) + (\beta_2 \times 0.6) + \varepsilon_3 \\
5.5 &= \beta_0 + (\beta_1 \times 29.1) + (\beta_2 \times 0.7) + \varepsilon_4 \\
9 &= \beta_0 + (\beta_1 \times 22) + (\beta_2 \times 0.8) + \varepsilon_5 \\
8.6 &= \beta_0 + (\beta_1 \times 29) + (\beta_2 \times 1.3) + \varepsilon_6 \\
12 &= \beta_0 + (\beta_1 \times 29.4) + (\beta_2 \times 1) + \varepsilon_7
\end{aligned}
$$

1.5. Multiple Linear Regression

1.5.1. Multiplicative model

$$growth = intercept + temp + nitro + temp \times nitro$$

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + ... + \epsilon_i$$

1.6. Multiple Linear Regression

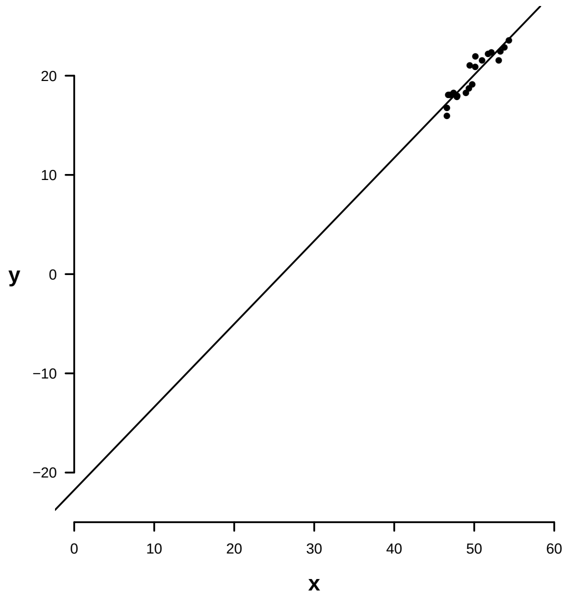
1.6.1. Multiplicative model

$$
\begin{aligned}
3 &= \beta_0 + (\beta_1 \times 22.7) + (\beta_2 \times 0.9) + (\beta_3 \times 22.7 \times 0.9) + \varepsilon_1 \\
2.5 &= \beta_0 + (\beta_1 \times 23.7) + (\beta_2 \times 0.5) + (\beta_3 \times 23.7 \times 0.5) + \varepsilon_2 \\
6 &= \beta_0 + (\beta_1 \times 25.7) + (\beta_2 \times 0.6) + (\beta_3 \times 25.7 \times 0.6) + \varepsilon_3 \\
5.5 &= \beta_0 + (\beta_1 \times 29.1) + (\beta_2 \times 0.7) + (\beta_3 \times 29.1 \times 0.7) + \varepsilon_4 \\
9 &= \beta_0 + (\beta_1 \times 22) + (\beta_2 \times 0.8) + (\beta_3 \times 22 \times 0.8) + \varepsilon_5 \\
8.6 &= \beta_0 + (\beta_1 \times 29) + (\beta_2 \times 1.3) + (\beta_3 \times 29 \times 1.3) + \varepsilon_6 \\
12 &= \beta_0 + (\beta_1 \times 29.4) + (\beta_2 \times 1) + (\beta_3 \times 29.4 \times 1) + \varepsilon_7
\end{aligned}
$$

2. Centering data

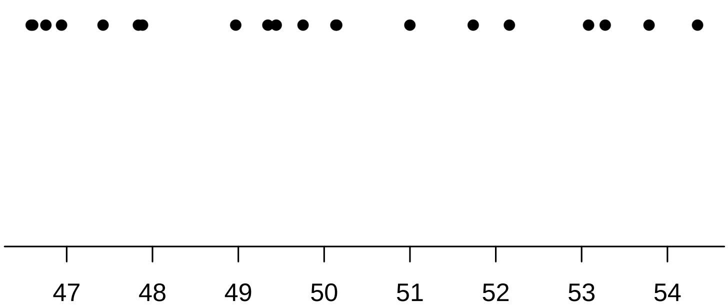
2.1. Multiple Linear Regression

2.1.1. Centering



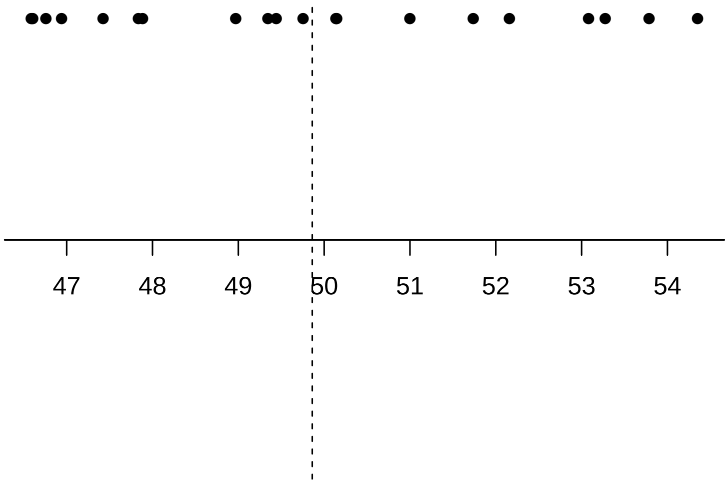
2.2. Multiple Linear Regression

2.2.1. Centering



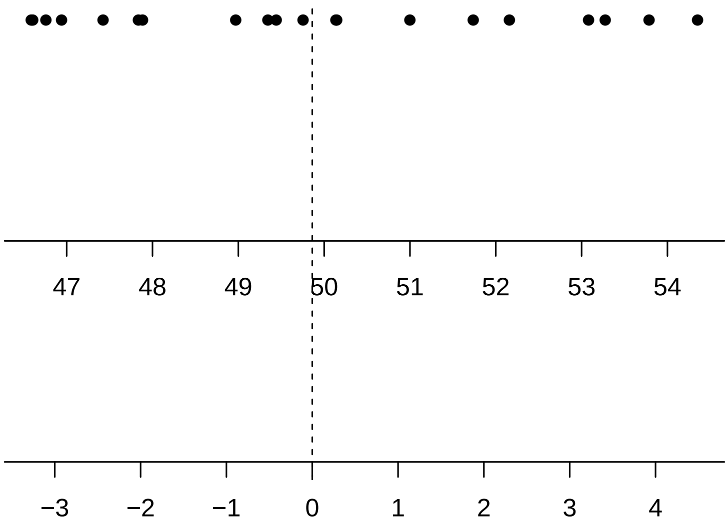
2.3. Multiple Linear Regression

2.3.1. Centering



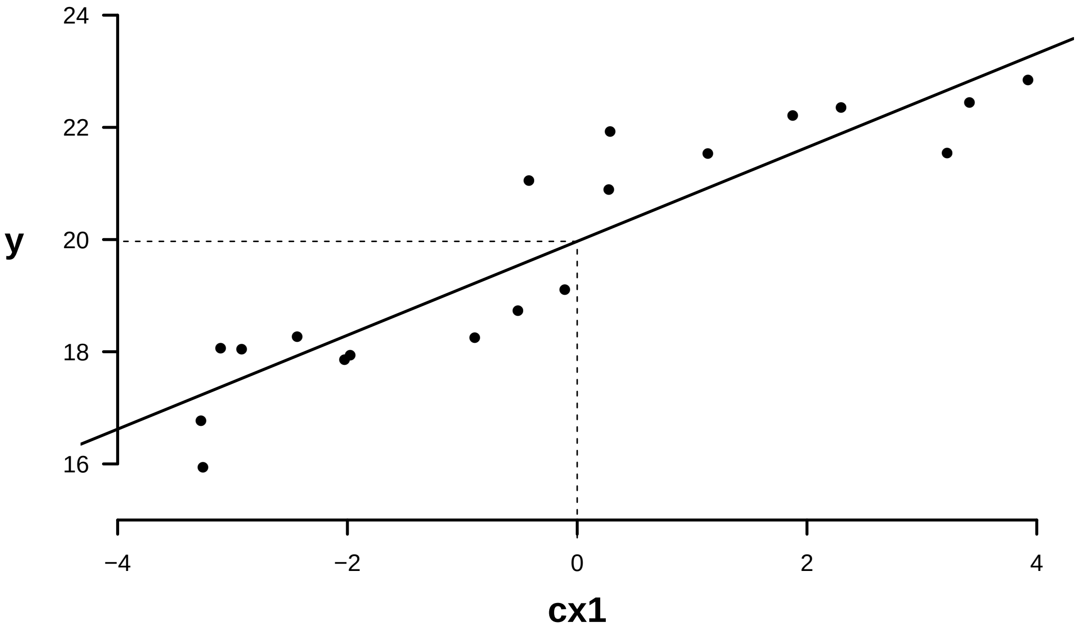
2.4. Multiple Linear Regression

2.4.1. Centering



2.5. Multiple Linear Regression

2.5.1. Centering

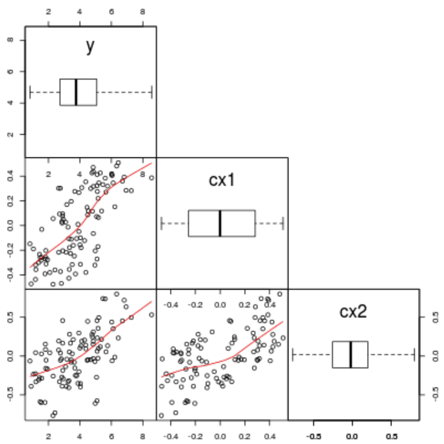


3. Assumptions

3.1. Multiple Linear Regression

3.1.1. Assumptions

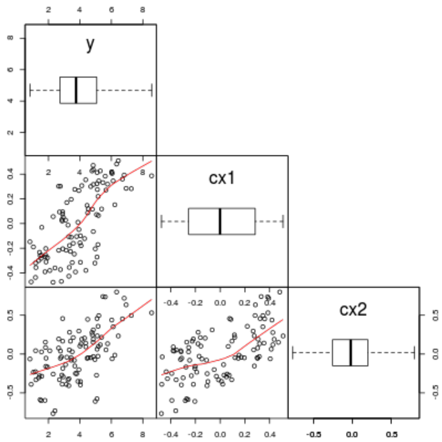
Normality, homog., linearity



3.2. Multiple Linear Regression

3.2.1. Assumptions

(multi)collinearity



3.3. Multiple Linear Regression

3.3.1. Variance inflation

Strength of a relationship

$$R^2$$

Strong when $R^2 \geq 0.8$

3.4. Multiple Linear Regression

3.4.1. Variance inflation

$$var.inf = \frac{1}{1 - R^2}$$

Collinear when $var.inf >= 5$

Some prefer > 3

3.5. Multiple Linear Regression

3.5.1. Assumptions

(multi)collinearity

```
library(car)
# additive model - scaled predictors
vif(lm(y ~ cx1 + cx2, data))
```

```
     cx1      cx2
1.743817 1.743817
```

3.6. Multiple Linear Regression

3.6.1. Assumptions

(multi)collinearity

```
library(car)
# additive model - scaled predictors
vif(lm(y ~ cx1 + cx2, data))
```

```
     cx1      cx2 
1.743817 1.743817 
```

```
# multiplicative model - raw predictors
vif(lm(y ~ x1 * x2, data))
```

```
       x1        x2     x1:x2 
 7.259729  5.913254 16.949468 
```

3.7. Multiple Linear Regression

3.7.1. Assumptions

```
# multiplicative model - raw predictors
vif(lm(y ~ x1 * x2, data))
```

```
       x1        x2     x1:x2 
 7.259729  5.913254 16.949468 
```

```
# multiplicative model - scaled predictors
vif(lm(y ~ cx1 * cx2, data))
```

```
     cx1      cx2  cx1:cx2 
1.769411 1.771994 1.018694 
```

4. Multiple linear models in R

4.1. Model fitting

Additive model

$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$

```
data.add.lm <- lm(y~cx1+cx2, data)
```

4.2. Model fitting

Additive model

$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$

```
data.add.lm <- lm(y~cx1+cx2, data)
```

Multiplicative model

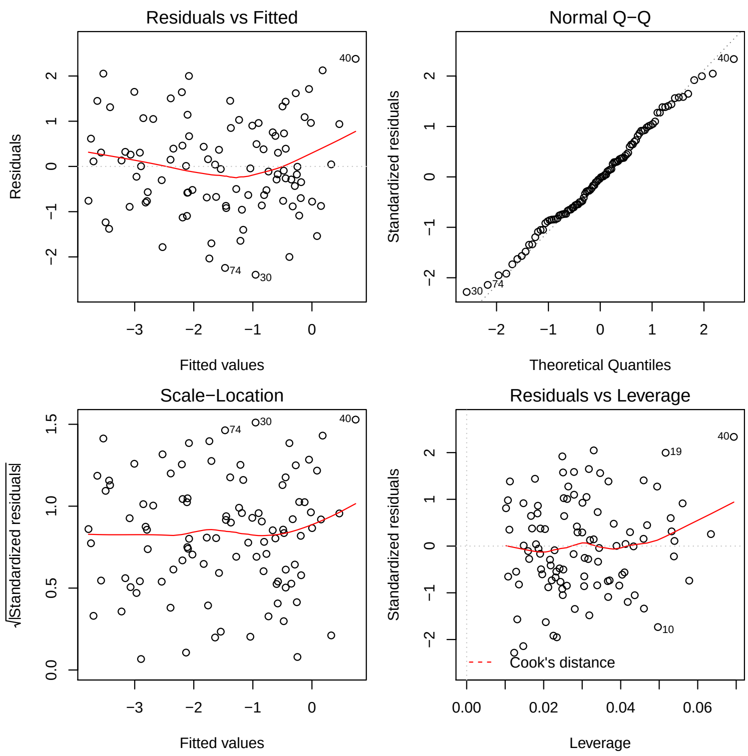
$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \epsilon_i$

```
data.mult.lm <- lm(y~cx1+cx2+cx1:cx2, data)
#OR
data.mult.lm <- lm(y~cx1*cx2, data)
```

4.3. Model evaluation

Additive model

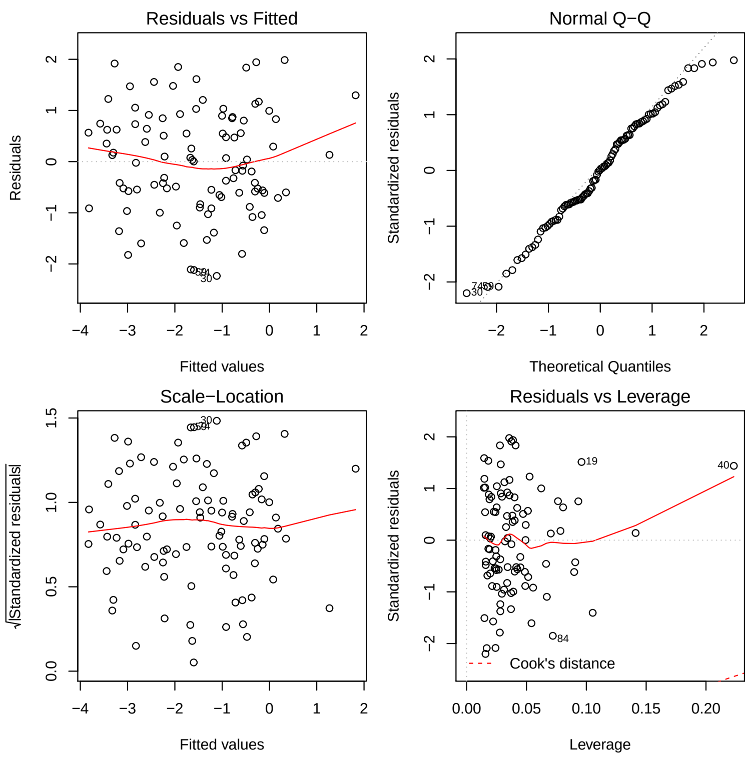
```
plot(data.add.lm)
```



4.4. Model evaluation

Multiplicative model

```
plot(data.mult.lm)
```



4.5. Model summary

Additive model

```
summary(data.add.lm)
```

```

Call:
lm(formula = y ~ cx1 + cx2, data = data)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.39418 -0.75888 -0.02463  0.73688  2.37938 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -1.5161     0.1055 -14.364  < 2e-16 ***
cx1           2.5749     0.4683   5.499  3.1e-07 ***
cx2          -4.0475     0.3734 -10.839  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.055 on 97 degrees of freedom
Multiple R-squared:  0.5567,	Adjusted R-squared:  0.5476 
F-statistic: 60.91 on 2 and 97 DF,  p-value: < 2.2e-16
```

4.6. Model summary

Additive model

```
confint(data.add.lm)
```

```
                2.5 %    97.5 %
(Intercept) -1.725529 -1.306576
cx1          1.645477  3.504300
cx2         -4.788628 -3.306308
```

4.7. Model summary

Multiplicative model

```
summary(data.mult.lm)
```

```

Call:
lm(formula = y ~ cx1 * cx2, data = data)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.23364 -0.62188  0.01763  0.80912  1.98568 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -1.6995     0.1228 -13.836  < 2e-16 ***
cx1           2.7232     0.4571   5.957 4.22e-08 ***
cx2          -4.1716     0.3648 -11.435  < 2e-16 ***
```

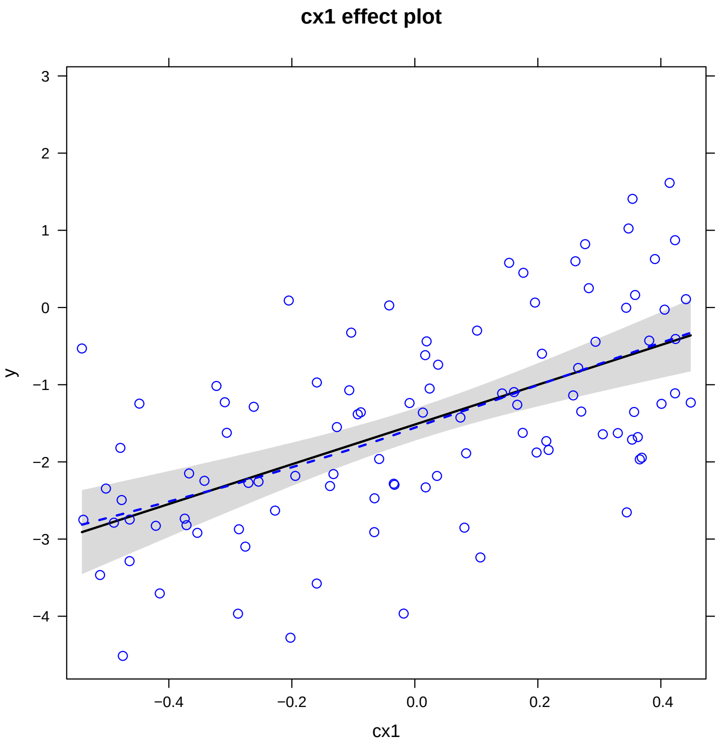
```
cx1:cx2       2.5283     0.9373   2.697  0.00826 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.023 on 96 degrees of freedom
Multiple R-squared:  0.588, Adjusted R-squared:  0.5751
F-statistic: 45.66 on 3 and 96 DF,  p-value: < 2.2e-16
```

4.8. Graphical summaries

Additive model

```
library(effects)
plot(effect("cx1",data.add.lm, partial.residuals=TRUE))
```



4.9. Graphical summaries

Additive model

```
library(effects)
library(ggplot2)
e <- effect("cx1",data.add.lm, xlevels=list(
      cx1=seq(-0.4,0.4, len=10)), partial.residuals=TRUE)
newdata <- data.frame(fit=e$fit, cx1=e$x, lower=e$lower,
      upper=e$upper)
resids <- data.frame(resid=e$partial.residuals.raw,
      cx1=e$data$cx1)
```

```
Error in data.frame(resid = e$partial.residuals.raw, cx1 = e$data$cx1): arguments imply differing number of r
```

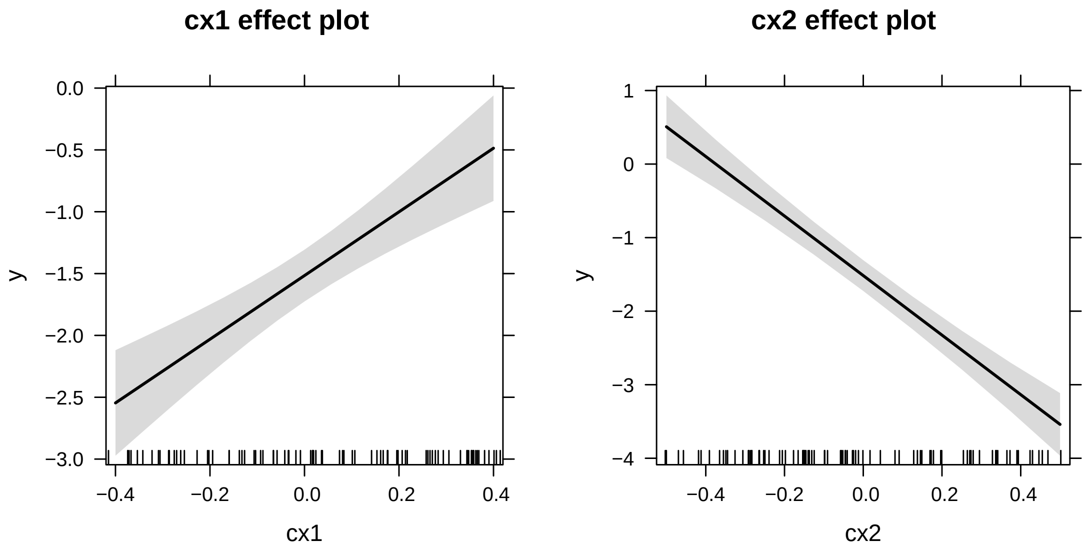
```
ggplot(newdata, aes(y=fit, x=cx1)) +
    geom_point(data=resids, aes(y=resid, x=cx1))+
    geom_ribbon(aes(ymin=lower, ymax=upper), fill='blue',
     alpha=0.2)+
    geom_line()+theme_classic()
```

```
Error in fortify(data): object 'resids' not found
```

4.10. *Graphical summaries*

Additive model

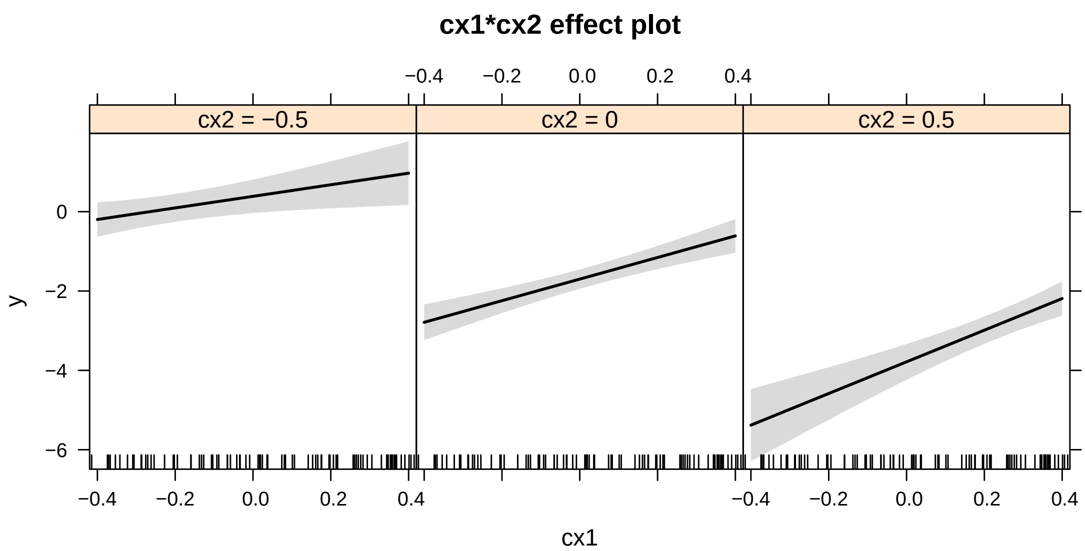
```
library(effects)
plot(allEffects(data.add.lm))
```



4.11. *Graphical summaries*

Multiplicative model

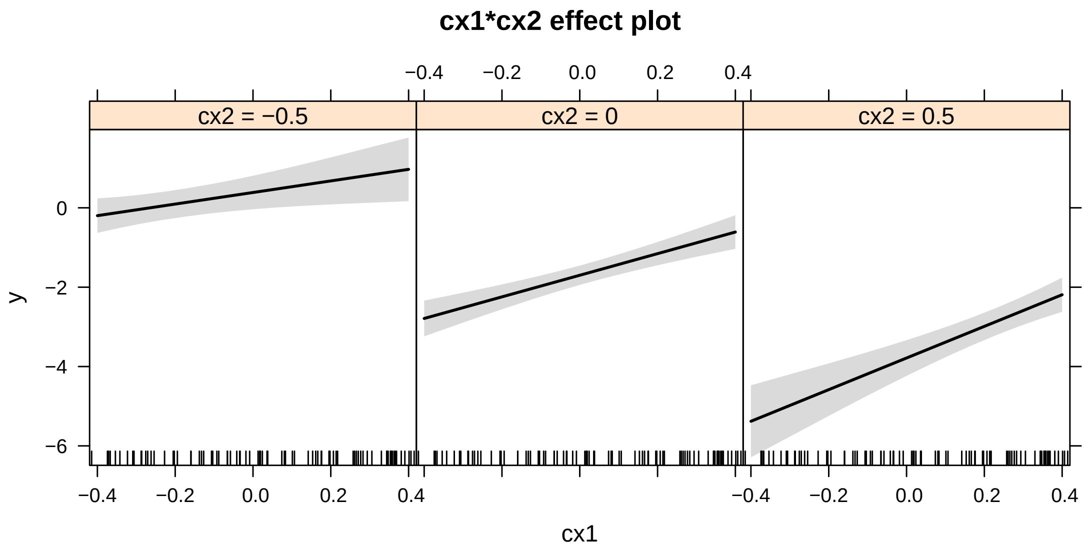
```
library(effects)
plot(allEffects(data.mult.lm))
```



4.12. *Graphical summaries*

Multiplicative model

```
library(effects)
plot(Effect(focal.predictors=c("cx1","cx2"),data.mult.lm))
```



5. Model selection

5.1. How good is a model?

"All models are wrong, but some are useful" George E. P. Box

5.1.1. Criteria

- R^2 - no
- Information criteria
  - AIC, AICc
  - penalize for complexity

5.2. Model selection

5.2.1. Candidates

```
AIC(data.add.lm, data.mult.lm)
```

```
             df      AIC
data.add.lm   4 299.5340
data.mult.lm  5 294.2283
```

```
library(MuMIn)
AICc(data.add.lm, data.mult.lm)
```

```
             df     AICc
data.add.lm   4 299.9551
data.mult.lm  5 294.8666
```

5.3. Model selection

5.3.1. Dredging

```
library(MuMIn)
data.mult.lm <- lm(y~cx1*cx2, data, na.action=na.fail)
dredge(data.mult.lm, rank="AICc", trace=TRUE)
```

```
0 : lm(formula = y ~ 1, data = data, na.action = na.fail)
1 : lm(formula = y ~ cx1 + 1, data = data, na.action = na.fail)
2 : lm(formula = y ~ cx2 + 1, data = data, na.action = na.fail)
3 : lm(formula = y ~ cx1 + cx2 + 1, data = data, na.action = na.fail)
7 : lm(formula = y ~ cx1 + cx2 + cx1:cx2 + 1, data = data, na.action = na.fail)

Global model call: lm(formula = y ~ cx1 * cx2, data = data, na.action = na.fail)
---
Model selection table
   (Int)     cx1    cx2 cx1:cx2 df   logLik  AICc delta weight
8 -1.699  2.7230 -4.172   2.528  5 -142.114 294.9  0.00  0.927
4 -1.516  2.5750 -4.047          4 -145.767 300.0  5.09  0.073
3 -1.516         -2.706          3 -159.333 324.9 30.05  0.000
1 -1.516                         2 -186.446 377.0 82.15  0.000
2 -1.516 -0.7399                 3 -185.441 377.1 82.27  0.000
Models ranked by AICc(x)
```

5.4. Multiple Linear Regression

5.4.1. Model averaging

```
library(MuMIn)
data.dredge<-dredge(data.mult.lm, rank="AICc")
model.avg(data.dredge, subset=delta<20)
```

```
Call:
model.avg(object = data.dredge, subset = delta < 20)

Component models:
'123' '12'

Coefficients:
       (Intercept)      cx1       cx2  cx1:cx2
full     -1.686125 2.712397 -4.162525 2.344227
subset   -1.686125 2.712397 -4.162525 2.528328
```

5.5. Multiple Linear Regression

5.5.1. Model selection

Or more preferable:

- identify 10-15 candidate models
- compare these via AIC (etc)

6. Worked Examples

6.1. Worked examples

```
loyn <- read.csv('../data/loyn.csv', strip.white=T)
head(loyn)
```

	ABUND	AREA	YR.ISOL	DIST	LDIST	GRAZE	ALT
1	5.3	0.1	1968	39	39	2	160
2	2.0	0.5	1920	234	234	5	60
3	1.5	0.5	1900	104	311	5	140
4	17.1	1.0	1966	66	66	3	160
5	13.8	1.0	1918	246	246	5	140
6	14.1	1.0	1965	234	285	3	130

6.2. Worked Examples

Question: what effects do fragmentation variables have on the abundance of forest birds

Linear model:

$$Abund_i = \beta_0 + \sum_{j=1:n}^{N} \beta_j X_{ji} + \varepsilon_i \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2)$$