Workshop 7.4a: Single factor ANOVA

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November 23, 2016

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1. Revision

1.1. Estimation

Y	X
3	0
2.5	1
6	2
5.5	3
9	4
8.6	5
12	6

$$
\begin{array}{ccccccc}
3.0 & = & \beta_0 \times 1 & + & \beta_1 \times 0 & + & \varepsilon_1 \\
2.5 & = & \beta_0 \times 1 & + & \beta_1 \times 1 & + & \varepsilon_1 \\
6.0 & = & \beta_0 \times 1 & + & \beta_1 \times 2 & + & \varepsilon_2 \\
5.5 & = & \beta_0 \times 1 & + & \beta_1 \times 3 & + & \varepsilon_3 \\
9.0 & = & \beta_0 \times 1 & + & \beta_1 \times 4 & + & \varepsilon_4 \\
8.6 & = & \beta_0 \times 1 & + & \beta_1 \times 5 & + & \varepsilon_5 \\
12.0 & = & \beta_0 \times 1 & + & \beta_1 \times 6 & + & \varepsilon_6
\end{array}
$$

1.2. Estimation

$$
\begin{array}{ccccccc}
3.0 & = & \beta_0 \times 1 & + & \beta_1 \times 0 & + & \varepsilon_1 \\
2.5 & = & \beta_0 \times 1 & + & \beta_1 \times 1 & + & \varepsilon_1 \\
6.0 & = & \beta_0 \times 1 & + & \beta_1 \times 2 & + & \varepsilon_2 \\
5.5 & = & \beta_0 \times 1 & + & \beta_1 \times 3 & + & \varepsilon_3 \\
9.0 & = & \beta_0 \times 1 & + & \beta_1 \times 4 & + & \varepsilon_4 \\
8.6 & = & \beta_0 \times 1 & + & \beta_1 \times 5 & + & \varepsilon_5 \\
12.0 & = & \beta_0 \times 1 & + & \beta_1 \times 6 & + & \varepsilon_6
\end{array}
$$

$$\underbrace{\begin{pmatrix}3.0\\2.5\\6.0\\5.5\\9.0\\8.6\\12.0\end{pmatrix}}_{\text{Response values}} = \underbrace{\begin{pmatrix}1&0\\1&1\\1&2\\1&3\\1&4\\1&5\\1&6\end{pmatrix}}_{\text{Model matrix}} \times \underbrace{\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix}}_{\text{Parameter vector}} + \underbrace{\begin{pmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{pmatrix}}_{\text{Residual vector}}$$

1.3. Matrix algebra

$$\underbrace{\begin{pmatrix}3.0\\2.5\\6.0\\5.5\\9.0\\8.6\\12.0\end{pmatrix}}_{\text{Response values}} = \underbrace{\begin{pmatrix}1&0\\1&1\\1&2\\1&3\\1&4\\1&5\\1&6\end{pmatrix}}_{\text{Model matrix}} \times \underbrace{\begin{pmatrix}\beta_0\\\beta_1\end{pmatrix}}_{\text{Parameter vector}} + \underbrace{\begin{pmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{pmatrix}}_{\text{Residual vector}}$$

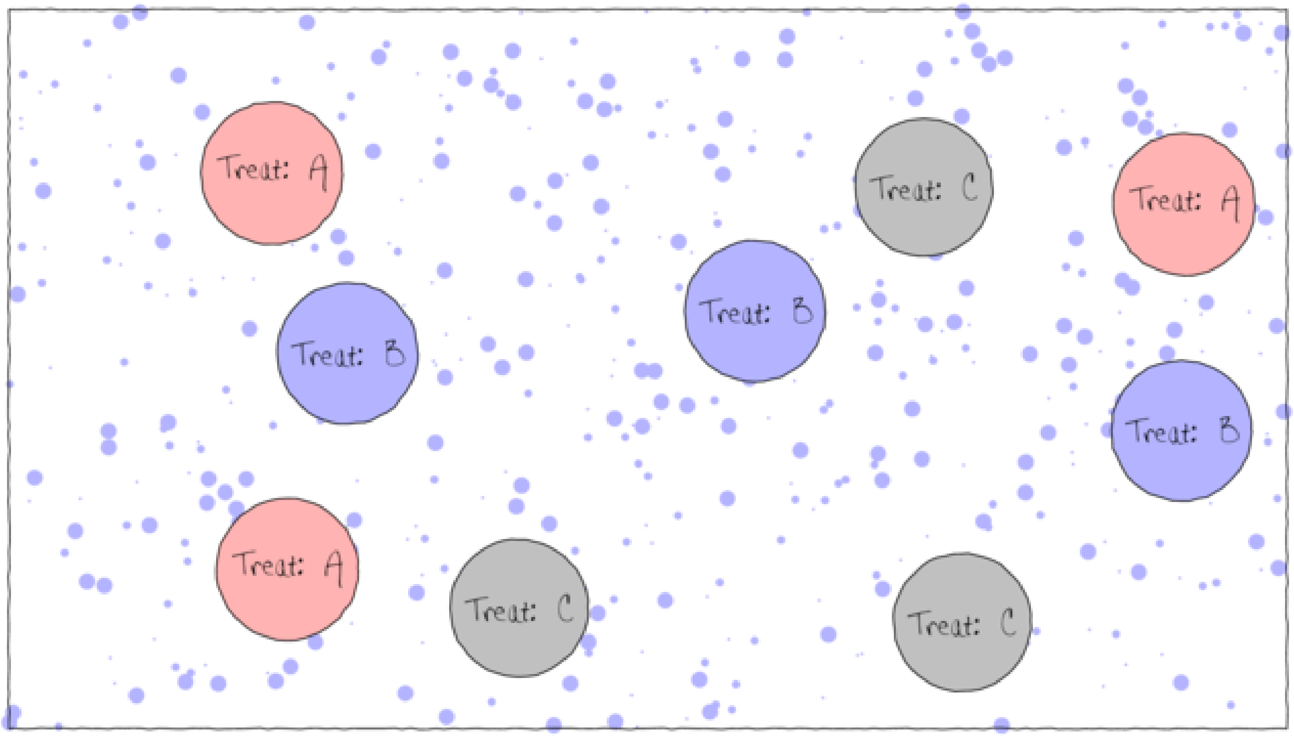
$$Y = X\beta + \epsilon$$

$$\hat{\beta} = (X'X)^{-1}X'Y$$

2. Anova Parameterization

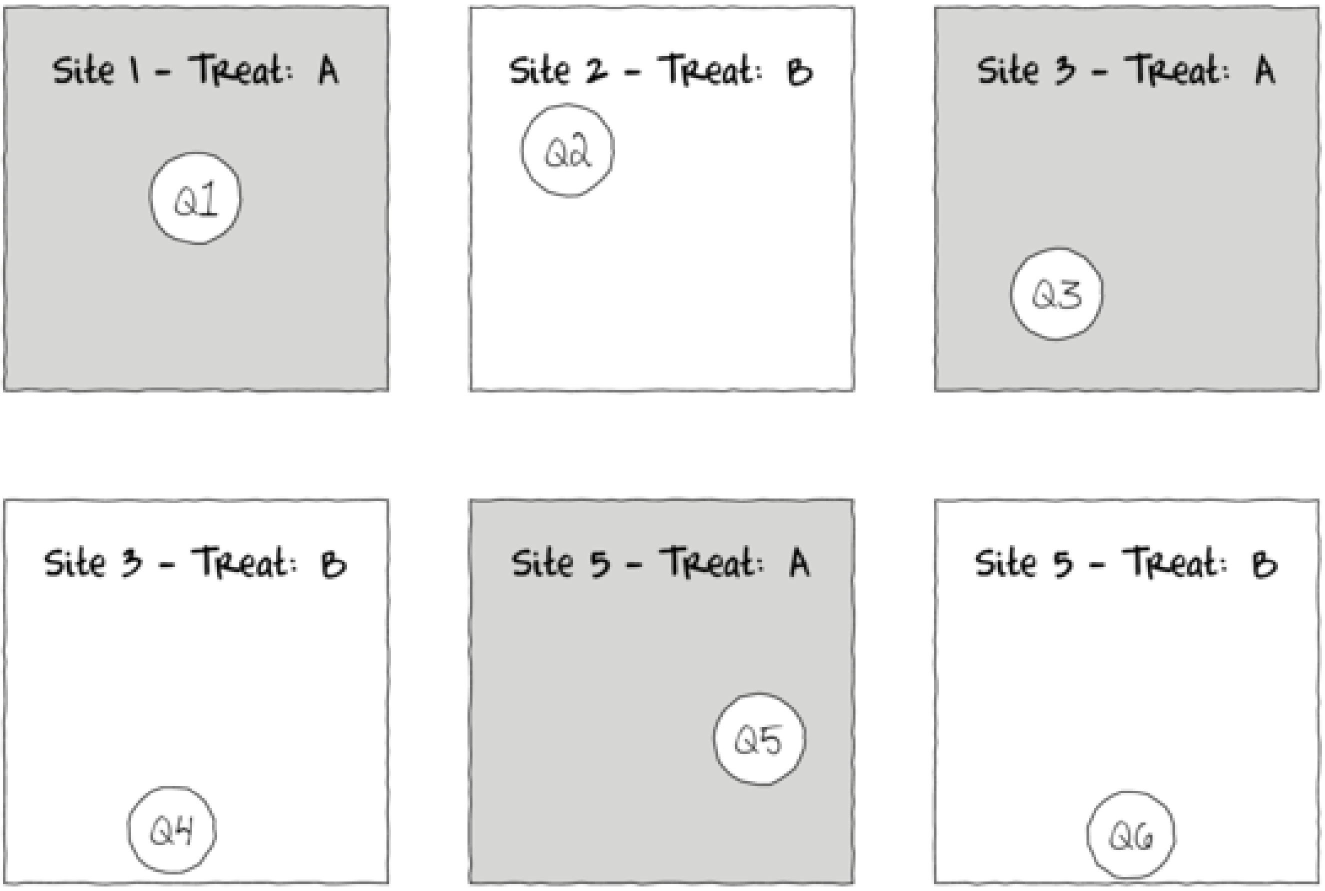
2.1. Simple ANOVA

Three treatments (One factor - 3 levels), three replicates



2.2. Simple ANOVA

Two treatments, three replicates



2.3. Categorical predictor

Y	A	dummy1	dummy2	dummy3
2	G1	1	0	0
3	G1	1	0	0
4	G1	1	0	0
6	G2	0	1	0
7	G2	0	1	0
8	G2	0	1	0
10	G3	0	0	1
11	G3	0	0	1
12	G3	0	0	1

$$y_{ij} = \mu + \beta_1(dummy_1)_{ij} + \beta_2(dummy_2)_{ij} + \beta_3(dummy_3)_{ij} + \varepsilon_{ij}$$

2.4. Overparameterized

$$y_{ij} = \mu + \beta_1(dummy_1)_{ij} + \beta_2(dummy_2)_{ij} + \beta_3(dummy_3)_{ij} + \varepsilon_{ij}$$

Y	A	Intercept	dummy1	dummy2	dummy3
2	G1	1	1	0	0
3	G1	1	1	0	0
4	G1	1	1	0	0
6	G2	1	0	1	0
7	G2	1	0	1	0
8	G2	1	0	1	0
10	G3	1	0	0	1
11	G3	1	0	0	1
12	G3	1	0	0	1

2.5. Overparameterized

$$y_{ij} = \mu + \beta_1(dummy_1)_{ij} + \beta_2(dummy_2)_{ij} + \beta_3(dummy_3)_{ij} + \varepsilon_{ij}$$

- three treatment groups
- **four** parameters to estimate
- need to **re-parameterize**

2.6. Categorical predictor

$$y_i = \mu + \beta_1(dummy_1)_i + \beta_2(dummy_2) + \beta_3(dummy_3)_i + \varepsilon_i$$

2.6.1. Means parameterization

$$y_i = \beta_1(dummy_1)_i + \beta_2(dummy_2)_i + \beta_3(dummy_3)_i + \varepsilon_{ij}$$

$$y_{ij} = \alpha_i + \varepsilon_{ij} \qquad \text{i} = \text{p}$$

2.7. Categorical predictor

2.7.1. Means parameterization

$$y_i = \beta_1(dummy_1)_i + \beta_2(dummy_2)_i + \beta_3(dummy_3)_i + \varepsilon_i$$

Y	A	dummy1	dummy2	dummy3
2	G1	1	0	0
3	G1	1	0	0
4	G1	1	0	0
6	G2	0	1	0
7	G2	0	1	0

```
 8  G2      0          1          0
10  G3      0          0          1
11  G3      0          0          1
12  G3      0          0          1
```

2.8. Categorical predictorDD

2.8.1. Means parameterization

	Y	A
1	2.00	G1
2	3.00	G1
3	4.00	G1
4	6.00	G2
5	7.00	G2
6	8.00	G2
7	10.00	G3
8	11.00	G3
9	12.00	G3

$$y_i = \alpha_1 D_{1i} + \alpha_2 D_{2i} + \alpha_3 D_{3i} + \varepsilon_i$$
$$y_i = \alpha_p + \varepsilon_i,$$

where p = number of levels of the factor and D = dummy variables

$$\begin{pmatrix}2\\3\\4\\6\\7\\8\\10\\11\\12\end{pmatrix} = \begin{pmatrix}1&0&0\\1&0&0\\1&0&0\\0&1&0\\0&1&0\\0&1&0\\0&0&1\\0&0&1\\0&0&1\end{pmatrix} \times \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} + \begin{pmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\\\varepsilon_7\\\varepsilon_8\\\varepsilon_9\end{pmatrix}$$

2.9. Categorical predictor

2.9.1. Means parameterization

Parameter	Estimates	Null Hypothesis
α_1^*	mean of group 1	H_0: $\alpha_1 = \alpha_1 = 0$
α_2^*	mean of group 2	H_0: $\alpha_2 = \alpha_2 = 0$
α_3^*	mean of group 3	H_0: $\alpha_3 = \alpha_3 = 0$

```
> summary(lm(Y~-1+A))$coef
```

```
    Estimate Std. Error   t value     Pr(>|t|)
AG1        3  0.5773503  5.196152 2.022368e-03
AG2        7  0.5773503 12.124356 1.913030e-05
AG3       11  0.5773503 19.052559 1.351732e-06
```

- but typically interested exploring **effects**

2.10. Categorical predictor

$$y_i = \mu + \beta_1(dummy_1)_i + \beta_2(dummy_2)_i + \beta_3(dummy_3)_i + \varepsilon_i$$

2.10.1. Effects parameterization

$$y_{ij} = \mu + \beta_2(dummy_2)_{ij} + \beta_3(dummy_3)_{ij} + \varepsilon_{ij}$$

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \qquad \mathrm{i} = \mathrm{p} - 1$$

2.11. Categorical predictor

2.11.1. Effects parameterization

$$y_i = \alpha + \beta_2(dummy_2)_i + \beta_3(dummy_3)_i + \varepsilon_i$$

Y	A	alpha	dummy2	dummy3
2	G1	1	0	0
3	G1	1	0	0
4	G1	1	0	0
6	G2	1	1	0
7	G2	1	1	0
8	G2	1	1	0
10	G3	1	0	1
11	G3	1	0	1
12	G3	1	0	1

2.12. Categorical predictor

2.12.1. Effects parameterization

	Y	A
1	2.00	G1
2	3.00	G1
3	4.00	G1
4	6.00	G2
5	7.00	G2
6	8.00	G2
7	10.00	G3
8	11.00	G3
9	12.00	G3

$y_i = \alpha + \beta_2 D_{2i} + \beta_3 D_{3i} + \varepsilon_i$
$y_i = \alpha_p + \varepsilon_i$,
where p = number of levels of the factor minus 1 and D = dummy variables

$$\begin{pmatrix} 2\\3\\4\\6\\7\\8\\10\\11\\12 \end{pmatrix} = \begin{pmatrix} 1&0&0\\1&0&0\\1&0&0\\1&1&0\\1&1&0\\1&1&0\\1&0&1\\1&0&1\\1&0&1 \end{pmatrix} \times \begin{pmatrix} \mu\\ \alpha_2\\ \alpha_3 \end{pmatrix} + \begin{pmatrix} \varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\\ \varepsilon_7\\ \varepsilon_8\\ \varepsilon_9 \end{pmatrix}$$

2.13. Categorical predictor

2.13.1. Treatment contrasts

Parameter	Estimates	Null Hypothesis
Intercept	mean of **control** group	H_0: $\mu = \mu_1 = 0$

Parameter	Estimates	Null Hypothesis
α_2^*	mean of group 2 minus mean of **control** group	H_0: $\alpha_2 = \alpha_2 = 0$
α_3^*	mean of group 3 minus mean of **control** group	H_0: $\alpha_3 = \alpha_3 = 0$

```
> contrasts(A) <-contr.treatment
> contrasts(A)
```

```
   2 3
G1 0 0
G2 1 0
G3 0 1
```

```
> summary(lm(Y~A))$coef
```

```
            Estimate Std. Error  t value     Pr(>|t|)
(Intercept)        3  0.5773503 5.196152 2.022368e-03
A2                 4  0.8164966 4.898979 2.713682e-03
A3                 8  0.8164966 9.797959 6.506149e-05
```

2.14. Categorical predictor

2.14.1. Treatment contrasts

Parameter	Estimates	Null Hypothesis
Intercept	mean of **control** group	H_0: $\mu = \mu_1 = 0$
α_2^*	mean of group 2 minus mean of **control** group	H_0: $\alpha_2 = \alpha_2 = 0$
α_3^*	mean of group 3 minus mean of **control** group	H_0: $\alpha_3 = \alpha_3 = 0$

```
> summary(lm(Y~A))$coef
```

```
            Estimate Std. Error  t value     Pr(>|t|)
(Intercept)        3  0.5773503 5.196152 2.022368e-03
A2                 4  0.8164966 4.898979 2.713682e-03
A3                 8  0.8164966 9.797959 6.506149e-05
```

2.15. Categorical predictor

2.15.1. User defined contrasts

User defined contrasts
Grp2 vs Grp3
Grp1 vs (Grp2 & Grp3)

	Grp1	Grp2	Grp3
α_2^*	0	1	-1
α_3^*	1	-0.5	-0.5

```
> contrasts(A) <- cbind(c(0,1,-1),c(1, -0.5, -0.5))
> contrasts(A)
```

```
   [,1] [,2]
G1    0  1.0
G2    1 -0.5
G3   -1 -0.5
```

2.16. Categorical predictor

2.16.1. User defined contrasts

- $p-1$ comparisons (contrasts)
- all contrasts must be **orthogonal**

2.17. Categorical predictor

2.17.1. Orthogonality

Four groups (A, B, C, D)

$p-1=3$ comparisons

1. A vs B :: A $>$ B
2. B vs C :: B $>$ C
3. A vs C ::

2.18. Categorical predictor

2.18.1. User defined contrasts

```
> contrasts(A) <- cbind(c(0,1,-1),c(1, -0.5, -0.5))
> contrasts(A)
```

```
   [,1] [,2]
G1    0  1.0
G2    1 -0.5
G3   -1 -0.5
```

$$\begin{aligned} 0 \times 1.0 &= 0 \\ 1 \times -0.5 &= -0.5 \\ -1 \times -0.5 &= 0.5 \\ \mathit{sum} &= 0 \end{aligned}$$

2.19. Categorical predictor

2.19.1. User defined contrasts

```
> contrasts(A) <- cbind(c(0,1,-1),c(1, -0.5, -0.5))
> contrasts(A)
```

```
   [,1] [,2]
G1    0  1.0
G2    1 -0.5
G3   -1 -0.5
```

```
> crossprod(contrasts(A))
```

```
     [,1] [,2]
[1,]    2  0.0
[2,]    0  1.5
```

```
> summary(lm(Y~A))$coef
```

```
            Estimate Std. Error   t value     Pr(>|t|)
(Intercept)        7  0.3333333 21.000000 7.595904e-07
A1                -2  0.4082483 -4.898979 2.713682e-03
A2                -4  0.4714045 -8.485281 1.465426e-04
```

2.20. Categorical predictor

2.20.1. User defined contrasts

```
> contrasts(A) <- cbind(c(1, -0.5, -0.5),c(1,-1,0))
> contrasts(A)
```

```
   [,1] [,2]
G1  1.0    1
G2 -0.5   -1
G3 -0.5    0
```

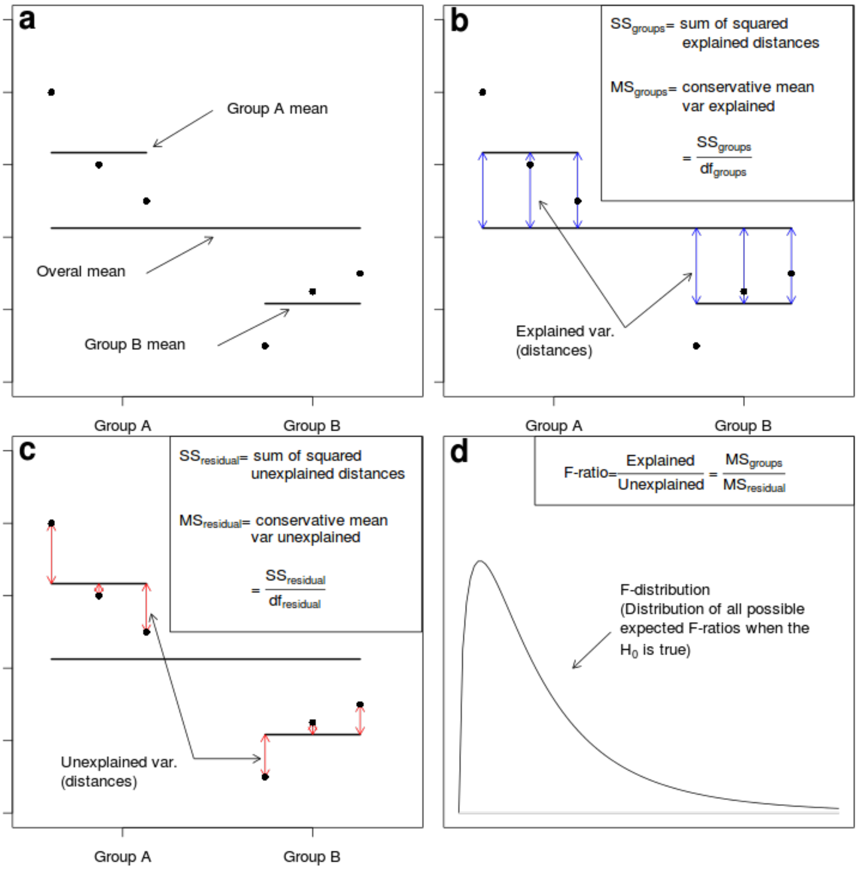
```
> crossprod(contrasts(A))
```

```
     [,1] [,2]
[1,]  1.5  1.5
[2,]  1.5  2.0
```

3. Partitioning of variance (ANOVA)

3.1. ANOVA

3.1.1. Partitioning variance



3.2. ANOVA

3.2.1. Partitioning variance

```
> anova(lm(Y~A))
```

```
Analysis of Variance Table

Response: Y
          Df Sum Sq Mean Sq F value    Pr(>F)
A          2     96      48      48 0.0002035 ***
Residuals  6      6       1
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

3.3. Categorical predictor

3.3.1. Post-hoc comparisons

No. of Groups	No. of comparisons	Familywise Type I error probability
3	3	0.14
5	10	0.40
10	45	0.90

3.4. Categorical predictor

3.4.1. Post-hoc comparisons

Bonferoni

```
> summary(lm(Y~A))$coef
```

```
            Estimate Std. Error   t value     Pr(>|t|)
(Intercept)        7  0.3333333 21.000000 7.595904e-07
A1                -8  0.9428090 -8.485281 1.465426e-04
A2                 4  0.8164966  4.898979 2.713682e-03
```

```
> 0.05/3
```

```
[1] 0.01666667
```

3.5. Categorical predictor

3.5.1. Post-hoc comparisons

Tukey's test

```
> library(multcomp)
> data.lm<-lm(Y~A)
> summary(glht(data.lm, linfct=mcp(A="Tukey")))
```

```
     Simultaneous Tests for General Linear Hypotheses

Multiple Comparisons of Means: Tukey Contrasts


Fit: lm(formula = Y ~ A)

Linear Hypotheses:
             Estimate Std. Error t value Pr(>|t|)
G2 - G1 == 0   4.0000     0.8165   4.899  0.00653 **
G3 - G1 == 0   8.0000     0.8165   9.798  < 0.001 ***
G3 - G2 == 0   4.0000     0.8165   4.899  0.00679 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Adjusted p values reported -- single-step method)
```

3.6. Assumptions

- Normality
- Homogeneity of variance
- Independence

- As for regression

4. Worked Examples

4.1. Worked Examples

```
> day <- read.csv('../data/day.csv', strip.white=T)
> head(day)
```

```
  TREAT BARNACLE
1  ALG1       27
2  ALG1       19
3  ALG1       18
4  ALG1       23
5  ALG1       25
6  ALG2       24
```

4.2. Worked Examples

Question: what effects do different substrate types have on barnacle recruitment

Linear model:

$$Barnacle_i = \mu + \alpha_j + \varepsilon_i \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2)$$

4.3. Worked Examples

```
> partridge <- read.csv('../data/partridge.csv', strip.white=T)
> head(partridge)
```

```
  GROUP LONGEVITY
1 PREG8        35
2 PREG8        37
3 PREG8        49
4 PREG8        46
5 PREG8        63
6 PREG8        39
```

```
> str(partridge)
```

```
'data.frame':   125 obs. of  2 variables:
 $ GROUP    : Factor w/ 5 levels "NONE0","PREG1",..: 3 3 3 3 3 3 3 3 3 3 ...
 $ LONGEVITY: int  35 37 49 46 63 39 46 56 63 65 ...
```

4.4. Worked Examples

Question: what effects does mating have on the longevity of male fruitflies

Linear model:

$$Longevity_i = \mu + \alpha_j + \varepsilon_i \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2)$$