Workshop 7.4b: Single factor ANOVA (Bayesian)

Murray Logan
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Table of contents

1. Revision

1.1. Estimation

Y	X
3	0
2.5	1
6	2
5.5	3
9	4
8.6	5
12	6

$$
\begin{array}{ccccccc}
3.0 & = & \beta_0 \times 1 & + & \beta_1 \times 0 & + & \varepsilon_1 \\
2.5 & = & \beta_0 \times 1 & + & \beta_1 \times 1 & + & \varepsilon_1 \\
6.0 & = & \beta_0 \times 1 & + & \beta_1 \times 2 & + & \varepsilon_2 \\
5.5 & = & \beta_0 \times 1 & + & \beta_1 \times 3 & + & \varepsilon_3 \\
9.0 & = & \beta_0 \times 1 & + & \beta_1 \times 4 & + & \varepsilon_4 \\
8.6 & = & \beta_0 \times 1 & + & \beta_1 \times 5 & + & \varepsilon_5 \\
12.0 & = & \beta_0 \times 1 & + & \beta_1 \times 6 & + & \varepsilon_6
\end{array}
$$

1.2. Estimation

Y	X
3	0
2.5	1
6	2
5.5	3

Y	X
9	4
8.6	5
12	6

$$
\begin{aligned}
3.0 &= \beta_0 \times 1 + \beta_1 \times 0 + \varepsilon_1 \\
2.5 &= \beta_0 \times 1 + \beta_1 \times 1 + \varepsilon_1 \\
6.0 &= \beta_0 \times 1 + \beta_1 \times 2 + \varepsilon_2 \\
5.5 &= \beta_0 \times 1 + \beta_1 \times 3 + \varepsilon_3 \\
9.0 &= \beta_0 \times 1 + \beta_1 \times 4 + \varepsilon_4 \\
8.6 &= \beta_0 \times 1 + \beta_1 \times 5 + \varepsilon_5 \\
12.0 &= \beta_0 \times 1 + \beta_1 \times 6 + \varepsilon_6
\end{aligned}
$$

$$
\underbrace{\begin{pmatrix} 3.0 \\ 2.5 \\ 6.0 \\ 5.5 \\ 9.0 \\ 8.6 \\ 12.0 \end{pmatrix}}_{\text{Response values}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{pmatrix}}_{\text{Model matrix}} \times \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}}_{\text{Parameter vector}} + \underbrace{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}}_{\text{Residual vector}}
$$

1.3. Matrix algebra

$$
\underbrace{\begin{pmatrix} 3.0 \\ 2.5 \\ 6.0 \\ 5.5 \\ 9.0 \\ 8.6 \\ 12.0 \end{pmatrix}}_{\text{Response values}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{pmatrix}}_{\text{Model matrix}} \times \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}}_{\text{Parameter vector}} + \underbrace{\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}}_{\text{Residual vector}}
$$

$$Y = X\beta + \epsilon$$

$$\hat{\beta} = (X'X)^{-1}X'Y$$

2. Anova Parameterization

2.1. Simple ANOVA

Three treatments (One factor - 3 levels), three replicates

2.2. Simple ANOVA

Two treatments, three replicates

2.3. Categorical predictor

Y	A	dummy1	dummy2	dummy3
2	G1	1	0	0
3	G1	1	0	0
4	G1	1	0	0
6	G2	0	1	0
7	G2	0	1	0
8	G2	0	1	0
10	G3	0	0	1
11	G3	0	0	1
12	G3	0	0	1

$$y_{ij} = \mu + \beta_1(dummy_1)_{ij} + \beta_2(dummy_2)_{ij} + \beta_3(dummy_3)_{ij} + \varepsilon_{ij}$$

2.4. Overparameterized

$$y_{ij} = \mu + \beta_1(dummy_1)_{ij} + \beta_2(dummy_2)_{ij} + \beta_3(dummy_3)_{ij} + \varepsilon_{ij}$$

Y	A	Intercept	dummy1	dummy2	dummy3
2	G1	1	1	0	0
3	G1	1	1	0	0
4	G1	1	1	0	0
6	G2	1	0	1	0
7	G2	1	0	1	0
8	G2	1	0	1	0
10	G3	1	0	0	1
11	G3	1	0	0	1
12	G3	1	0	0	1

2.5. Overparameterized

$$y_{ij} = \mu + \beta_1(dummy_1)_{ij} + \beta_2(dummy_2)_{ij} + \beta_3(dummy_3)_{ij} + \varepsilon_{ij}$$

- three treatment groups
- **four** parameters to estimate
- need to **re-parameterize**

2.6. Categorical predictor

$$y_i = \mu + \beta_1(dummy_1)_i + \beta_2(dummy_2)_i + \beta_3(dummy_3)_i + \varepsilon_i$$

2.6.1. Effects parameterization

$$y_{ij} = \mu + \beta_2(dummy_2)_{ij} + \beta_3(dummy_3)_{ij} + \varepsilon_{ij}$$

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \qquad \mathrm{i = p - 1}$$

2.7. Categorical predictor

2.7.1. Means parameterization

$$y_i = \alpha + \beta_2(dummy_2)_i + \beta_3(dummy_3)_i + \varepsilon_i$$

Y	A	alpha	dummy2	dummy3
2	G1	1	0	0
3	G1	1	0	0
4	G1	1	0	0
6	G2	1	1	0
7	G2	1	1	0
8	G2	1	1	0
10	G3	1	0	1
11	G3	1	0	1
12	G3	1	0	1

2.8. Categorical predictor

2.8.1. Means parameterization

0.3

	Y	A
1	2.00	G1
2	3.00	G1
3	4.00	G1
4	6.00	G2
5	7.00	G2
6	8.00	G2
7	10.00	G3
8	11.00	G3
9	12.00	G3

0.6

$y_i = \alpha + \beta_2 D_{2i} + \beta_3 D_{3i} + \varepsilon_i y_i = \alpha_p + \varepsilon_i$, where $p =$ number of levels of the factor minus 1and $D =$ dummy variables

$$\begin{pmatrix}2\\3\\4\\6\\7\\8\\10\\11\\12\end{pmatrix} = \begin{pmatrix}1&0&0\\1&0&0\\1&0&0\\1&1&0\\1&1&0\\1&1&0\\1&0&1\\1&0&1\\1&0&1\end{pmatrix} \times \begin{pmatrix}\mu\\\alpha_2\\\alpha_3\end{pmatrix} + \begin{pmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\\\varepsilon_7\\\varepsilon_8\\\varepsilon_9\end{pmatrix}$$

2.9. Categorical predictor

2.9.1. Treatment contrasts

Parameter	Estimates	Null Hypothesis
Intercept	mean of **control** group	H_0: $\mu = \mu_1 = 0$
α_2^*	mean of group 2 minus mean of **control** group	H_0: $\alpha_2 = \alpha_2 = 0$
α_3^*	mean of group 3 minus mean of **control** group	H_0: $\alpha_3 = \alpha_3 = 0$

```
> contrasts(A) <-contr.treatment
> contrasts(A)
```

```
   2 3
G1 0 0
G2 1 0
G3 0 1
```

```
> summary(lm(Y~A))$coef
```

```
            Estimate Std. Error  t value     Pr(>|t|)
(Intercept)        3  0.5773503 5.196152 2.022368e-03
A2                 4  0.8164966 4.898979 2.713682e-03
A3                 8  0.8164966 9.797959 6.506149e-05
```

2.10. Categorical predictor

2.10.1. Treatment contrasts

Parameter	Estimates	Null Hypothesis
Intercept	mean of **control** group	H_0: $\mu = \mu_1 = 0$
α_2^*	mean of group 2 minus mean of **control** group	H_0: $\alpha_2 = \alpha_2 = 0$
α_3^*	mean of group 3 minus mean of **control** group	H_0: $\alpha_3 = \alpha_3 = 0$

```
> summary(lm(Y~A))$coef
```

```
             Estimate Std. Error  t value     Pr(>|t|)
(Intercept)        3  0.5773503 5.196152 2.022368e-03
A2                 4  0.8164966 4.898979 2.713682e-03
A3                 8  0.8164966 9.797959 6.506149e-05
```

2.11. Assumptions

- Normality
- Homogeneity of variance
- Independence

- As for regression

2.12. The model

Y	A	dummy1	dummy2	dummy3
2	G1	1	0	0
3	G1	1	0	0
4	G1	1	0	0
6	G2	0	1	0
7	G2	0	1	0
8	G2	0	1	0
10	G3	0	0	1
11	G3	0	0	1
12	G3	0	0	1

$$y_{ij} \sim N(\mu_{ij}, \sigma^2)$$
$$\mu_{ij} = \beta_0 + \beta_j X_{ij}$$

Priors:

$$\beta_0 \sim N(0, \sigma^2_{Int})$$
$$\beta_j \sim N(0, \sigma^2_{Treat})$$
$$\sigma^2 \sim cauchy(0, 4)$$

3. Worked Examples

3.1. Worked Examples

```
> day <- read.csv('../data/day.csv', strip.white=T)
> head(day)
```

```
  TREAT BARNACLE
1  ALG1       27
2  ALG1       19
```

```
3   ALG1        18
4   ALG1        23
5   ALG1        25
6   ALG2        24
```

3.2. Worked Examples

Question: what effects do different substrate types have on barnacle recruitment

Linear model:

$$
\begin{aligned}
Barnacle_{ij} &\sim N(\mu_{ij}, \sigma^2) \\
\mu_{ij} &= \beta_0 + \beta_j Treat_{ij}
\end{aligned}
$$

$$
\begin{aligned}
\beta_0 &\sim N(0, \sigma^2_{Int}) \\
\beta_j &\sim N(0, \sigma^2_{Treat}) \\
\sigma^2 &\sim cauchy(0, 4)
\end{aligned}
$$